GROUP ACTIONS ON CENTRAL SIMPLE ALGEBRAS

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Abstract. Let G be a group, F a field, and A a finite-dimensional central simple algebra over F on which G acts by F-algebra automorphisms. We study the ideals and subalgebras of A which are preserved by the group action. Let V be the unique simple module of A. We show that V is a projective representation of G and $A \cong \mathrm{End}_D(V)$ makes V into a projective representation. We then prove that there is a natural one-to-one correspondence between G-invariant D-submodules of V and invariant left (and right) ideals of A.

Under the assumption that V is irreducible, we show that an invariant (unital) subalgebra must be a simply embedded semisimple subalgebra. We introduce induction of G-algebras. We show that each invariant subalgebras is induced from a simple H-algebra for some subgroup H of finite index and obtain a parametrization of the set of invariant subalgebras in terms of induction data. We then describe invariant central simple subalgebras. For F algebraically closed, we obtain an entirely explicit classification of the invariant subalgebras. Furthermore, we show that the set of invariant subalgebras is finite if G is a finite group. Finally, we consider invariant subalgebras when V is a continuous projective representation of a topological group G. We show that if the connected component of the identity acts irreducibly on V, then all invariant subalgebras are simple. We then apply our results to obtain a particularly nice solution to the classification problem when G is a compact connected Lie group and $F = \mathbf{C}$.

1. Introduction

Let G be a group, F a field, and V a finite-dimensional F-vector space on which G acts by F-linear automorphisms. A fundamental problem in representation theory is to classify the G-invariant subspaces of V, in other words, to determine those subspaces of V which inherit a G-action from V. For the case when G is a compact group and $F = \mathbf{C}$, this question has been answered completely. The representation can be decomposed canonically into a direct sum of subrepresentations $V = U_1 \oplus \cdots \oplus U_m$, where each U_i is the direct sum of n_i copies of an irreducible representation V_i and the V_i's are pairwise nonisomorphic. The G-invariant subspaces of U_i are parametrized by subspaces of $\mathbf{C}^{n_i}$ while the subrepresentations of V are direct sums of subrepresentations of the U_i's which may be chosen independently. As long as a decomposition of V into irreducible components is given explicitly (which may be very difficult in practice), this classification is also entirely explicit.

Let us now replace the vector space V with a finite-dimensional F-algebra A. We suppose further that A is a G-algebra, i.e G acts on A by F-algebra automorphisms, so that the G-action is well-behaved with respect to ring multiplication.

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The natural analogue of the problem considered above is to determine those G-invariant subspaces of A which have significance in terms of the multiplicative structure of A. In particular, we would like to classify the G-invariant ideals (left, right, and two-sided) and subalgebras. These are just special cases of the general problem of understanding the multiplication of subrepresentations of A. If M and N are two subrepresentations of A, then MN, the F-linear span of the set $\{mn \mid m \in M, n \in N\}$, is also G-invariant. We thus obtain a multiplication on the set of subrepresentations of A. Invariant ideals and algebras are now easily expressed in terms of this multiplication; an invariant left ideal is a subrepresentation I such that $AI \subset I$, an invariant subalgebra is a subrepresentation B such that $BB \subset B$, and so on.

These problems are much more difficult than the classification of G-invariant subspaces. It is unreasonable to expect to find a way of determining G-invariant ideals and subalgebras that works for all A, even for G compact and $F = \mathbf{C}$. Indeed, if we let G act trivially on A, then this result would give a uniform way of classifying ideals and subalgebras. It is thus necessary to limit the class of algebras under consideration.

In this paper, we restrict attention to central simple algebras over F. Our initial motivation for doing so came from a problem in solid state physics. The study of G-actions on real and complex central simple algebras is important in understanding how physical properties such as conductivity, elasticity, and piezoelectricity of a composite material depend on the properties of its constituents. These physical characteristics are described by elements of a symmetric tensor space $\mathrm{Sym}^2(T)$, where T is a certain real representation of the rotation group $SO(n)$. In general, a property of a composite depends heavily on the microstructure, i.e. the arrangement of the component materials. Let $M \subset \mathrm{Sym}^2(T)$ be the set of all possible values of a fixed property for composites made with their constituents taken in prescribed volume fractions. Typically, M is the closure of an open set in $\mathrm{Sym}^2(T)$ and may be described by a system of inequalities, so that away from the boundary of M, it is possible to make any desired small change in the property by varying the microgeometry. However, in certain unusual situations, some of the inequalities become equations, determining a proper closed submanifold E in which M is locally closed. The submanifold E and also the equations defining E are called exact relations for the property. The variability of the property with microstructure is thus drastically reduced when an exact relation is present. Recent work of Grabovsky, Milton, and Sage has shown how to classify exact relations in terms of the multiplication of $SO(n)$-subrepresentations in the endomorphism algebra $\mathrm{End}_{\mathbf{R}}(T)$; in particular, invariant algebras and ideals of this central simple algebra have great physical significance [G, GS, GMS].

Let A be a central simple algebra over F, and suppose G acts on A by F-algebra automorphisms. In the first part of this paper, we show that A is isomorphic to the algebra of D-endomorphisms of a projective representation V of G, where D is a certain central division algebra. We then prove that there is a natural one-to-one correspondence between G-invariant D-submodules of V and invariant left (and right) ideals of A. Indeed, we show that if G is compact and A is the endomorphism algebra of a complex representation, then the parametrization of invariant left and right ideals of A is the same as the classical parametrization of invariant subspaces of V described above. In particular, V is irreducible if and only if there are no

invariant proper left (right) invariant ideals, and V is multiplicity free if and only if there are a finite number of left (right) invariant ideals.

In the second part of the paper, we turn to the much more complicated problem of understanding unital invariant subalgebras of A under the additional hypothesis that V is an irreducible projective representation. We show that an invariant subalgebra B must be a simply embedded semisimple subalgebra; this means that both B and its centralizer in A must be direct products of isomorphic simple algebras. We then introduce induction of G-algebras. We show that each invariant subalgebras is induced from a simple H-algebra for some subgroup H of finite index and obtain a parametrization of the set of invariant subalgebras in terms of induction data. We then describe invariant central simple subalgebras.

Combining these two results, we obtain an entirely explicit classification of the invariant subalgebras for F algebraically closed. This classification shows that the set of invariant subalgebras of A encodes complicated information about G and V, involving both how V can be expressed as an induced representation $\mathrm{Ind}_H^G(W)$ and how W can be factored into the tensor product of projective representations. It should be observed that for $F = \mathbf{C}$ and G finite, knowing the character table of G does not suffice to determine all invariant subalgebras. In fact, even in the simplest case where V is a primitive representation, the character table of a covering group of G is needed to find all invariant subalgebras. When V is primitive, we show that the only nonunital invariant subalgebra is $\{0\}$. Finally, we prove that for G finite and F algebraically closed, the set of invariant subalgebras is finite, and we describe how finiteness fails in the general case.

In the final section of the paper, we consider invariant subalgebras when V is a continuous projective representation of a topological group G. We show that if the connected component of the identity acts irreducibly on V, then all invariant subalgebras are simple. We then apply our results to obtain a theorem of Etingof giving a particularly elegant solution to the classification problem when G is a compact connected Lie group and $F = \mathbf{C}$. In fact, suppose G is semisimple and simply connected, say $G = G_1 \times \cdots \times G_n$ with each G_i simple. The representation V is then isomorphic to $V_1 \otimes \cdots \otimes V_n$, for some irreducible representations V_i of G_i. We show that the G-invariant subalgebras of A are parametrized by the subsets J of $\{i \mid V_i \neq \mathbf{C}\}$ via $J \mapsto \bigotimes_{j\in J} \mathrm{End}_{\mathbf{C}}(V_j)$ and that the only nonunital invariant subalgebra is $\{0\}$. In particular, if G is simple, the invariant subalgebras are $\mathbf{C}$ and A.

We have also obtained results on the general problem of multiplication of subrepresentations in central simple algebras when G is a compact, simply reducible group. This means that g and g^{-1} are conjugate for all $g \in G$ (so that all G-modules are self-dual) and G is multiplicity-free, i.e. if V and W are irreducible, then each isotypic component of $V \otimes W$ is irreducible. (The most familiar examples of simply reducible groups are S_3, S_4, the quaternion group, $SU(2)$, and $SO(3)$.) However, since the proofs use quite different techniques, these results will appear in another paper [S].

It is a great pleasure to thank Yury Grabovsky for first bringing these problems to my attention and for explaining their importance in physics. I would also like to thank Daniel Allcock for several helpful comments and Pavel Etingof for letting me use his unpublished result on invariant subalgebras of compact connected Lie groups.

2. Preliminaries and Invariant Ideals

Let A be a finite-dimensional central simple algebra over the field F, and let V be a simple (left) A-module. The module V is unique up to isomorphism and is a finite-dimensional vector space over F. By Schur's Lemma, the ring $D = \mathrm{End}_A(V)$ is a central division algebra, and V is naturally a left D-module. It is well-known that A is isomorphic to $\mathrm{End}_D(V)$, and from now on, we assume without loss of generality that $A = \mathrm{End}_D(V)$.

It is easy to construct examples of central simple algebras on which the group G acts by F-algebra automorphisms. Recall that a mapping $\rho : G \to GL(V)$ is called a projective representation of G over F if $\rho(1) = 1_V$ and if there exists $\alpha : G \times G \to F^*$ such that $\rho(xy) = \alpha(x, y)\rho(x)\rho(y)$ for all $x, y \in G$. (Equivalently, we can view a projective representation as a homomorphism $G \to PGL(V)$.) The map α is a 2-cocycle. Let $\bar{g}$ be the basis vector corresponding to $g \in G$ in the twisted group algebra $F^\alpha G$. A projective α-representation is just an $F^\alpha G$-module via $\bar{g}v = \rho(g)(v)$, and we also use this notation. (For linear representations, we just write gv.) The map $\pi : G \to GL(A)$ then makes A into a (linear) representation of G with $(\pi(g)f)(v) = \rho(g)(f(\rho(g)^{-1}(v))$ for all $g \in G$, $f \in A$, and $v \in V$. Moreover, the linear map $\pi(g)$ is in fact an algebra automorphism. It turns out that all central simple algebras on which G acts via algebra automorphisms are of this type.

Proposition 2.1. *Suppose that G acts on $A = End_D(V)$ by F-algebra automorphisms, i.e. A is a representation of G via a homomorphism $G \xrightarrow{\pi} Aut(A)$. Then V is a projective representation of G determined up to projective equivalence, and the G-action on A is the natural action induced by the projective G-action on V.*

Proof. Any automorphism of A is inner by the Skolem-Noether theorem. Hence, we obtain a function $\hat{\rho} : G \to A^\times \subset GL(V)$ such that $\pi(g)(a) = \hat{\rho}(g)a\hat{\rho}(g)^{-1}$ for all $g \in G$ and $a \in A$. Since $\pi(1) = 1_A$, we have $\hat{\rho}(1) \in Z(A)^\times = F^*$. Setting $\rho(g) = \hat{\rho}(g)/\hat{\rho}(1)$ gives $\rho(1) = 1_V$. Also, the equation $\pi(gh) = \pi(g)\pi(h)$ implies that $\rho(gh)\rho(h)^{-1}\rho(g)^{-1}$ is central and therefore a nonzero multiple of the identity. It follows that (V, ρ) is a projective representation of G giving rise to π. $\square$

We now recall the ideal structure of A. Let $\mathcal{S}(V)$ denote the set of D-subspaces of V partially ordered by inclusion. This poset is in fact a complete lattice, with the greatest lower bound and least upper bound of a collection of subspaces given by their intersection and sum respectively. Similarly, the sets $\mathcal{L}(A)$ and $\mathcal{R}(A)$ of left and right ideals of A are complete lattices. It will be convenient to work with the dual lattice $\mathcal{L}(A)^*$ of left ideals under reverse inclusion (and with the supremum and infimum reversed). If L is a D-submodule of V, we define the annihilator and coannihilator of L by $\mathrm{Ann}(L) = \{f \in A \mid f(L) = 0\}$ and $\mathrm{Coann}(L) = \{f \in A \mid f(V) \subset L\}$; these are respectively left and right ideals of A. We then have the well-known fact that all left and right ideals of A are of this form.

Proposition 2.2. *The maps $\mathcal{S}(V) \xrightarrow{Ann} \mathcal{L}(A)^*$ and $\mathcal{S}(V) \xrightarrow{Coann} \mathcal{R}(A)$ are isomorphisms of complete lattices. The inverses are given by $I \mapsto \bigcap_{f \in I} Ker(f)$ and $J \mapsto \sum_{f \in J} f(V)$, where $I \in \mathcal{L}(A)$ and $J \in \mathcal{R}(A)$.*

Remark. In matrix language, this simply says that a left ideal consists of all matrices (with respect to some basis depending on the ideal) with zeroes in given columns while a right ideal consists of all matrices with zeros in given rows.

Let $\mathcal{S}_G(V) \subset \mathcal{S}(V)$ be the complete sublattice of all D-subspaces of V preserved by the G-action on V. Similarly, we define the complete sublattices $\mathcal{L}_G(A) \subset \mathcal{L}(A)$ and $\mathcal{R}_G(A) \subset \mathcal{R}(A)$ of G-invariant left and right ideals of A. It is natural to conjecture that the sublattices $\mathcal{L}_G(A)$ and $\mathcal{R}_G(A)$ are just the images of $\mathcal{S}_G(V)$ under the above isomorphisms, i.e. invariant left and right ideals are annihilators and coannihilators respectively of subrepresentations of V. This is indeed the case.

Theorem 2.3. *The restrictions of the maps Ann and Coann define isomorphisms of complete lattices* $\mathcal{S}_G(V) \stackrel{Ann}{\to} \mathcal{L}_G(A)^*$ *and* $\mathcal{S}_G(V) \stackrel{Coann}{\longrightarrow} \mathcal{R}_G(A)$.

Proof. In order to prove the first isomorphism, it suffices to show that $\mathrm{Ann}(\mathcal{S}_G(V)) \subset \mathcal{L}_G(A)^*$ and $\mathrm{Ann}^{-1}(\mathcal{L}_G(A)^*) \subset \mathcal{S}_G(V)$. If L is a subrepresentation of V and $f \in \mathrm{Ann}(L)$, then $(g \cdot f)(v) = \bar{g}(f(\bar{g}^{-1}(v))) = \bar{g}(0) = 0$ for all $g \in G$ and $v \in L$. Thus, $\mathrm{Ann}(L)$ is G-invariant. Conversely, if I is an invariant left ideal and $v \in \mathrm{Ann}^{-1}(I) = \bigcap_{f \in I} \mathrm{Ker}(f)$, then we also have $v \in \bigcap_{f \in I} \mathrm{Ker}(g \cdot f)$. Since $\rho(g)$ is bijective, this gives $f(\bar{g}^{-1}v) = 0$ for all $g \in G$ and $f \in I$. It follows that $\mathrm{Ann}^{-1}(I)$ is G-invariant.

The proof for invariant right ideals is similar. □

Remarks. 1. Since A is simple, the only two-sided ideals are $\{0\}$ and A which are of course G-invariant. However, it is a general fact that if B is an arbitrary G-algebra on which G acts by inner automorphisms, then all two-sided ideals are G-invariant. Indeed, if I is a two-sided ideal and the action of g on B is given by conjugation by $b_g \in B^\times$, then $gI = b_g I b_g^{-1} \subset I$.

2. Suppose that F is algebraically closed and V is a completely reducible linear representation of G, say $V \cong n_1V_1 \oplus \cdots \oplus n_mV_m$ where the V_i's are pairwise non-isomorphic irreducible representations. Then the G-invariant left (and right) ideals of $\mathrm{End}_F(V)$ are parametrized by $\prod_{i=1}^m \{\text{subspaces of } F^{n_i}\}$.

This theorem allows us to characterize certain properties of representations in terms of the associated endomorphism algebras.

Corollary 2.4.
1. *The projective representation V is irreducible if and only if $End_F(V)$ has no proper invariant one-sided ideals.*
2. *Let D be a central division algebra, and suppose V is a D-module on which G acts (projectively) by D-linear automorphisms. Then V is D-irreducible (i.e. has no G-invariant D-submodules) if and only if $End_D(V)$ has no proper invariant one-sided ideals.*
3. *Suppose that F is an infinite field and V is completely reducible. Then V is multiplicity free if and only if $End_F(V)$ has a finite number of invariant one-sided ideals.*

Proof. The first two statements are clear from the theorem. The last follows from the second remark and the fact that for an infinite field, a vector space has an infinite number of subspaces if and only if it has dimension larger than one. □

It is worth noting that in spite of the strong connection between subrepresentations and invariant ideals, the group action on a subrepresentation does not determine the action on the corresponding left and right invariant ideals or vice versa.

If B is a semisimple (finite-dimensional) algebra on which G acts by inner automorphisms, this theorem can be used to determine the invariant ideals of B. Let $B = B_1 \oplus \cdots \oplus B_s$ where the simple component B_i can be viewed as $\mathrm{End}_{D_i}(V_i)$ where

D_i is a finite-dimensional division algebra over F and V_i is a finite-dimensional D_i-module. By the first remark, the two-sided ideal B_i is invariant and is thus a simple algebra on which G acts by inner automorphisms. Since left and right ideals of B are just direct sums of left and right ideals of B_i, we obtain the following corollary:

Corollary 2.5. *The maps* $\prod_{i=1}^s \mathcal{S}_G(V_i) \to \mathcal{L}_G(B)^*$ *and* $\prod_{i=1}^s \mathcal{S}_G(V_i) \to \mathcal{R}_G(B)$ *given by* $(L_1, \dots, L_s) \mapsto \bigoplus_{i=1}^s Ann(L_i)$ *and* $(L_1, \dots, L_s) \mapsto \bigoplus_{i=1}^s Coann(L_i)$ *respectively are isomorphisms of complete lattices.*

It is also possible to obtain analogous results for certain spaces of homomorphisms between two representations of G. Let V and W be two finite-dimensional linear representations over the division algebra D (which as above is finite-dimensional over F) on which G acts by D-linear automorphisms. The F-vector space $\mathrm{Hom}_D(V, W)$ is a representation of G via the action $(g \cdot f)(v) = g(f(g^{-1}v))$; moreover, it has the structure of an $(\mathrm{End}_D(W), \mathrm{End}_D(V))$-bimodule. Let L be a D-submodule of V, and define the annihilator of L by $\mathrm{Ann}(L) = \{f \in \mathrm{Hom}_D(V, W) \mid f(L) = 0\}$. This is a (left) $\mathrm{End}_D(W)$-submodule, and all $\mathrm{End}_D(W)$-submodules of $\mathrm{Hom}_D(V, W)$ are of this form. In fact, if we denote the lattice of $\mathrm{End}_D(W)$-submodules of $\mathrm{Hom}_D(V, W)$ by $\mathcal{L}(\mathrm{Hom}_D(V, W))$, the map $\mathcal{S}(V) \overset{\mathrm{Ann}}{\to} \mathcal{L}(\mathrm{Hom}_D(V, W))^*$ is an isomorphism of complete lattices. Similarly, for M a D-submodule of W, we have the (right) $\mathrm{End}_D(V)$-submodule $\mathrm{Coann}(M) = \{f \in \mathrm{Hom}_D(V, W) \mid f(V) \subset L\}$, and denoting the lattice of $\mathrm{End}_D(V)$-submodules of $\mathrm{Hom}_D(V, W)$ by $\mathcal{R}(\mathrm{Hom}_D(V, W))$, we obtain the isomorphism of complete lattices $\mathcal{S}(W) \overset{\mathrm{Coann}}{\longrightarrow} \mathcal{R}(\mathrm{Hom}_D(V, W))$. Again, we can characterize the sublattices $\mathcal{L}_G(\mathrm{Hom}_D(V, W))$ and $\mathcal{R}_G(\mathrm{Hom}_D(V, W))$ of G-invariant submodules in terms of these isomorphisms.

Theorem 2.6. *The restrictions of the maps Ann and Coann define isomorphisms of complete lattices* $\mathcal{S}_G(V) \overset{Ann}{\to} \mathcal{L}_G(Hom_D(V, W))^*$ *and* $\mathcal{S}_G(W) \overset{Coann}{\longrightarrow} \mathcal{R}_G(Hom_D(V, W))$.

Proof. The proof is similar to the proof of Theorem 2.3. □

Remark. The only sub-bimodules of $\mathrm{Hom}_D(V, W)$ are $\{0\}$ and $\mathrm{Hom}_D(V, W)$, which are G-invariant.

3. Invariant Subalgebras

We now turn our attention to subalgebras of the algebra $A = \mathrm{End}_D(V)$ which are preserved by the group action. (All subalgebras will be assumed to contain 1 unless otherwise specified.) Invariant subalgebras are much more difficult to understand than invariant ideals, and in general, invariant subalgebras can be very badly behaved. For example, if we let G act trivially on $\mathrm{End}_F(V)$, then every subalgebra is invariant. This means that if V has dimension n, then $\mathrm{End}_F(V)$ contains every n-dimensional F-algebra as an invariant subalgebra. Moreover, it is not even true that the ring of invariants A^G need be semisimple, if G is infinite or G is finite with the characteristic of F dividing $|G|$ [M]. We will therefore need to place additional restrictions on the G-algebra A.

Definition. A central simple G-algebra over F is called *G-simple* if the associated projective representation V is irreducible.

We assume from now on that A is G-simple. Note that under this hypothesis, the possible pathologies involving A^G are avoided, since by Schur's lemma, A^G is a division algebra.

We will show that all G-invariant subalgebras of A are semisimple with a very special structure. Indeed, we will give a complete classification when F is algebraically closed. As a first step, we have the following result:

Proposition 3.1. *Let B be an invariant subalgebra of A. Then B is semisimple, and the Wedderburn components of B are all isomorphic as F-algebras. Moreover, if U is any simple B-submodule of V, then for each $g \in G$, $\bar{g}U$ is also a simple B-submodule, and any simple B-module is isomorphic to some $\bar{g}U$.*

Proof. The inclusion of B in A makes the A-module V into a B-module. Let U be a simple B-submodule of V; for example, take U to be a B-submodule of minimal dimension as an F-vector space. Consider the translate $\bar{g}U$ for $g \in G$. Note that the G-invariance of B implies that

$$b\bar{g}(u) = \bar{g}\bar{g}^{-1}b\bar{g}(u) = \bar{g}(g^{-1} \cdot b)(u) \in \bar{g}U \tag{1}$$

for all $b \in B$ and $u \in U$. Here, we have used the fact that $\overline{g^{-1}} = \alpha(g, g^{-1})\bar{g}^{-1}$, where α is the cocycle defined by (V, ρ). Thus, $\bar{g}U$ is a B-submodule of V. Moreover, $\bar{g}U$ is simple, since the same argument shows that if W is a submodule of $\bar{g}U$, then $\overline{g^{-1}}W$ is a submodule of U. The sum $\sum_{g\in G} \bar{g}U$ is evidently a nonzero G-invariant subspace of V, and by irreducibility, $V = \sum_{g\in G} \bar{g}U$. Thus, V is a semisimple B-module, and we can choose $g_1, \dots, g_l \in G$ such that $V = \oplus_{i=1}^{l} \bar{g}_i U$.

Let $u_1, \dots, u_k$ be an F-basis for U. The map $B \to \oplus_{i=1}^{l} k(\overline{g_i}U)$ given by $b \mapsto (b\overline{g_1}u_1, \dots, b\overline{g_1}u_k, \dots, b\overline{g_l}u_1, \dots, b\overline{g_l}u_k)$ is a B-homomorphism. If b is in the kernel, then b kills an F-basis of V, and since $b \in A \subseteq \mathrm{End}_F(V)$, we have $b = 0$; hence, the map is injective. This shows that B is a semisimple F-algebra, and any simple B-module is isomorphic to $\bar{g}U$ for some $g \in G$. The simple components of B are of the form $\mathrm{End}_{D_g}(\bar{g}U)$, where $D_g = \mathrm{End}_B(\bar{g}U)$. To complete the proof, it suffices to verify that $\mathrm{End}_{D_g}(\bar{g}U)$ is isomorphic to $\mathrm{End}_{D'}(U)$, where $D' = D_1$.

We first show that the division algebras D' and D_g are isomorphic via the map $d \to \bar{g}d\bar{g}^{-1}$. Using the formula for the B-action on $\bar{g}U$ given in (1), we have $\bar{g}d\bar{g}^{-1}(b\bar{g}u) = \bar{g}d\bar{g}^{-1}(\bar{g}(g^{-1} \cdot b)u)) = \bar{g}d((g^{-1} \cdot b)u) = \bar{g}(g^{-1} \cdot b)d(u) = b\bar{g}d(u) = b\bar{g}d\bar{g}^{-1}(\bar{g}u)$ for all $d \in D'$ and $u \in U$, so that $\bar{g}d\bar{g}^{-1} \in D_g$. It is clear that this is an F-algebra homomorphism. In fact, it is an isomorphism with inverse map $D_g \to D'$ given by $\hat{d} \mapsto \overline{g^{-1}}\hat{d}\overline{g^{-1}}^{-1}$. This follows since $\bar{g}\overline{g^{-1}} \in F^*$ and elements of D' and D_g are F-linear.

Now suppose $f \in \mathrm{End}_{D'}(U)$. The F-map $\bar{g}f\bar{g}^{-1} : \bar{g}U \to \bar{g}U$ is D_g linear as$\bar{g}f\bar{g}^{-1}(\bar{g}d\bar{g}^{-1}(\bar{g}u)) = \bar{g}f(du) = \bar{g}df(u) = \bar{g}d\bar{g}^{-1}(\bar{g}f\bar{g}^{-1}(\bar{g}u))$. Thus, we have an F-algebra homomorphism $\mathrm{End}_{D'}(U) \to \mathrm{End}_{D_g}(\bar{g}U)$, $f \mapsto \bar{g}f\bar{g}^{-1}$, which is in fact an isomorphism with inverse $\hat{f} \mapsto \overline{g^{-1}}\hat{f}\overline{g^{-1}}^{-1}$. $\square$

Corollary 3.2. *The invariant subalgebra B is simple if and only if any for any simple B-submodule U of V, the B-modules U and $\bar{g}U$ are isomorphic for all $g \in G$.*

Although the proposition places significant restrictions on the structure of a G-invariant subalgebra, it turns out that the subalgebra must satisfy a much more stringent condition which depends on the ambient algebra A. For the time being, let $B = B_1 \oplus \ldots \oplus B_l$ be an arbitrary semisimple subalgebra of $A = \mathrm{End}_D(V)$ where the B_i's are simple F-algebras with corresponding simple modules W_i'. Note that $V = \oplus_{i=1}^{l} m_i' W_i'$ with positive multiplicities m_i' (or else $1_{B_i}(V) = 0$ for some i, contradicting the fact that the central primitive idempotent 1_{B_i} is a nonzero element

of A). The subalgebra B consists of D-linear maps, so V can also be viewed as a (B, D^{op})-bimodule or equivalently as a $B \otimes_F D$-module. Since D is central simple, $B \otimes D$ is semisimple with Wedderburn components $B_1 \otimes D, \dots, B_l \otimes D$, and we can write $V = \oplus_{i=1}^l m_i W_i$, where the W_i's are the simple $B \otimes D$-modules, with $W_i = W_i' \otimes D$ isomorphic to a minimal left ideal of $B_i \otimes_F D$. Again, each m_i is nonzero. In fact, we can say more.

Lemma 3.3. *Let V_i' and V_i be the isotypic B and $B \otimes_F D$-submodules of V for W_i' and W_i respectively. Then $V_i' = V_i$ and $m_i = m_i' / \dim_F D$. In particular, V_i' is a D-submodule of V.*

Proof. Recall that V_i' and V_i are the one-eigenspaces of the central primitive idempotents 1_{B_i} and $1_{B_i} \otimes 1_D$. Since these are the same maps on V, we have $V_i' = V_i$ for all i. Also $\dim W_i = (\dim D)(\dim W_i')$, so $m_i' = m_i \dim D$. □

Definition. A semisimple subalgebra B of $A = \mathrm{End}_D(V)$ is called *symmetrically embedded* if the Wedderburn components of B are all isomorphic as F-algebras and if the simple B modules appearing in V have the same multiplicity m', i.e. if $m' = m_1' = \cdots = m_l'$. (It is equivalent to replace either condition with the analogous statement involving $B \otimes D$.)

More explicitly, an element $b \in B$ acts on each copy of W_i in the same way, so is represented by a block diagonal matrix (in $M_{\dim_D V}(D^{op})$) with $m_1 + \cdots + m_l$ blocks, m_i of which consist of the $\dim_D W_i \times \dim_D W_i$ matrix corresponding to $b|_{W_i}$. If B is symmetrically embedded, then the blocks are all the same size and for each i, the matrix for $b|_{W_i}$ appears m times. Note that this implies that $\dim_D V = ml \dim_D W_i = ml \dim_F W_i'$ for any i.

It is clear from the above lemma that whether a subalgebra satisfies the above property does not depend on the central division algebra D. Indeed, we have:

Proposition 3.4. *Suppose that V is a module for two central division algebras D and D'. If B is a semisimple subalgebra of both $A = End_D(V)$ and $A' = End_{D'}(V)$, then B is symmetrically embedded in A if and only if it is symmetically embedded in A'.*

Proof. Since both A and A' are subalgebras of $\mathrm{End}_F(V)$, we can assume without loss of generality that $D' = F$, and this case follows immediately from the lemma. □

In order to see the importance of symmetrically embedded subalgebras, we need to recall some information about centralizers of semisimple subalgebras of central simple algebras. Let $Z_A(B)$ denote the centralizer in A of the subalgebra B. We call B a *Howe subalgebra* if it equals its double centralizer $Z_A Z_A(B)$ and say that the pair $(B, Z_A(B))$ is a *dual pair*. A strong version of the Double Centralizer Theorem states that if B is semisimple, then $Z_A(B)$ is also semisimple and B is a Howe subalgebra [J, Theorem 4.10]. In other words, the mapping $B \mapsto Z_A(B)$ provides a duality operator on the set of semisimple subalgebras of A. It is possible to calculate the Wedderburn structure of $Z_A(B)$ by an argument due to Moeglin, Vignéras, and Waldspurger [MVW, p.12]. Note that $f \in \mathrm{End}_{B \otimes_F D}(V)$ if and only if f is a D-linear map which commutes with the action of B, i.e. if and only if $f \in Z_A(B)$. Using the decomposition $V = \oplus_{i=1}^l m_i W_i$ of V into simple $B \otimes_F D$-submodules, it is immediate that $\mathrm{End}_{B \otimes D}(V) \cong \oplus_{i=1}^l \mathrm{End}_{B \otimes D}(m_i W_i) \cong \oplus_{i=1}^l M_{m_i}(D_i)$, where $D_i = \mathrm{End}_{B \otimes_F D}(W_i)$ is a division algebra over F. Since $W_i = W_i' \otimes D$, D_i is canonically isomorphic to $\mathrm{End}_B(W_i')$. Summing up, we have:

Theorem 3.5. *There is a duality on the set of semisimple subalgebras of A given by $B \mapsto Z_A(B)$ which preserves the number of Wedderburn components of the subalgebras. Moreover, if $V \cong \oplus_{i=1}^l m_i W_i$ is the decomposition of V into simple $B \otimes_F D$-modules and D_i is the division algebra $End_B(W_i') = End_{B\otimes_F D}(W_i)$, then $Z_A(B) \cong \oplus_{i=1}^l M_{m_i}(D_i)$.*

There are also two maps from a semisimple subalgebra to the set of self-dual (i.e. commutative) semisimple subalgebras, given by $B \mapsto Z(B)$, the center of B, and $B \mapsto Z_0(B)$, the F-linear span of the central primitive idempotents of B. These are respectively the largest and smallest self-dual subalgebras with the same central primitive idempotents as B. It is clear that both maps are constant on dual pairs. With the notation of the theorem, $Z(B) \cong \oplus_{i=1}^l Z(D_i)$ and $Z_0(B) \cong F^l$.

We can now reformulate the concept of a symmetrically embedded subalgebra in terms of centralizers.

Proposition 3.6. *A semisimple subalgebra B is symmetrically embedded in A if and only if both B and $Z_A(B)$ are direct sums of isomorphic simple F-algebras.*

Proof. Suppose B is symmetrically embedded. By definition, the Wedderburn components of B are all isomorphic. The Jacobson density theorem implies that $B \otimes_F D \cong \mathrm{End}_{D_i}(W_i)$ for all i, and by the structure theorem for simple Artinian algebras, the D_i's are all isomorphic as F-algebras. Since the multiplicities of the W_i's in V are the same, it follows from Theorem 3.5 that $Z_A(B)$ is a direct sum of isomorphic simple F-algebras.

Conversely, suppose that both B and $Z_A(B)$ have isomorphic Wedderburn components. Then $M_{m_i}(D_i) \cong M_{m_j}(D_j)$ for all i and j, and so the m_i's are equal by the structure theorem for simple Artinian algebras. Thus, B is symmetrically embedded in A. □

Next, we need an easy, but important lemma on centralizers of invariant subalgebras.

Lemma 3.7. *Let R be a G-algebra, and S a G-invariant subalgebra. Then the centralizer $Z_R(S)$ is also an invariant subalgebra. In particular, the center of S $Z(S) = Z_S(S)$ is an invariant subalgebra.*

Proof. This follows immediately from the fact that $(g \cdot z)s = g \cdot (z(g^{-1} \cdot s)) = g \cdot ((g^{-1} \cdot s)z) = s(g \cdot z)$ for all $g \in G$, $s \in S$, and $z \in Z_R(S)$. □

Combining the lemma with Propositions 3.1 and 3.6, we obtain the theorem:

Theorem 3.8. *Let B be an invariant subalgebra of A. Then B is symmetrically embedded in A.*

We now describe a fundamental construction of invariant subalgebras. We will then show that all invariant subalgebras are of this type and obtain a classification of them.

We first need to introduce induction of G-algebras. Let H be a subgroup of finite index in G, and suppose that C is an H-algebra. We show how to define a natural G-algebra structure on $\mathrm{Ind}_H^G(C)$ making Ind_H^G into a functor from the category of H-algebras into the category of G-algebras.

Proposition 3.9. *There is a unique G-algebra structure on $Ind_H^G(C) = FG \otimes_{FH} C$ extending the H-algebra $1 \otimes C$ such that distinct G-translates of $1 \otimes C$ annihilate each*

other. If $\{g_1, \dots, g_n\}$ is a left transversal for H in G, then the algebra multiplication is given by $(g_i \otimes b)(g_j \otimes b') = \delta_{ij}(g_i \otimes bb')$ for $b, b' \in C$. As F-algebras, $\mathrm{Ind}_H^G(C)$ is isomorphic to C^n. Furthermore, this definition makes Ind_H^G into a functor from the category of H-algebras into the category of G-algebras.

Proof. Uniqueness is clear. To show existence, recall that the coinduced representation $\mathrm{Hom}_{FH}(FG, C)$ (with G acting by $(g\cdot f)(x) = f(xg)$ for $x, g \in G$) is isomorphic to $\mathrm{Ind}_H^G(C)$ via the map $\phi \mapsto \sum_{i=1}^n g_i \otimes \phi(g_i^{-1})$. If ϕ and ψ are FH-linear, then $\phi\psi$ is as well, since $(\phi\psi)(hy) = (h\phi(y))(h\psi(y)) = h \cdot (\phi\psi(y))$ for $h \in H$ and $y \in FG$. Thus, pointwise multiplication makes $\mathrm{Hom}_{FH}(FG, C)$ into a G-algebra; translating the multiplication back to $\mathrm{Ind}_H^G(C)$ gives the desired formula. The elements $g_i \otimes 1$ are pairwise orthogonal central idempotents summing to the identity element in $\mathrm{Ind}_H^G(C)$, which is thereby isomorphic to $\bigoplus_{i=1}^n (g_i \otimes C) \cong C^n$ as F-algebras.

Now let C' be another H-algebra, and let $\psi : C \to C'$ be an H-algebra map. It is immediate that the G-module map $\mathrm{Ind}_H^G(\psi)$ is also an algebra homomorphism. (Under the above identifications, it is just $\psi \oplus \dots \oplus \psi : C^n \to (C')^n$.) Thus, Ind_H^G is a functor. ∎

Remarks. 1. If H does not have finite index in G, then $\mathrm{Ind}_H^G(B)$ is a nonunital G-algebra. Indeed, the coinduced representation is still a G-algebra, and $\mathrm{Ind}_H^G(B)$ is isomorphic to the nonunital subalgebra of FH-maps which are finitely supported modulo H.

2. If B is an interior H-algebra, i.e. H acts on B by inner automorphisms, then there is another way of defining an induced G-algebra originally introduced by Puig. These two concepts are quite different. Indeed, the underlying G-module in Puig's construction is not $\mathrm{Ind}_H^G(B)$, but instead $\mathrm{Ind}_H^G(B) \otimes_{FH} FG$. The resulting F-algebra structure is isomorphic to $M_n(B)$ instead of B^n [T, §16].

It is easy to check that this functor satisfies the usual properties of induction.

Proposition 3.10. *Let H be a subgroup of G of finite index, and suppose that C and C' are H-algebras.*

1. *$\mathrm{Ind}_H^G(C \oplus C') \cong \mathrm{Ind}_H^G(C) \oplus \mathrm{Ind}_H^G(C')$ and $\mathrm{Ind}_H^G(C \cap C') = \mathrm{Ind}_H^G(C) \cap \mathrm{Ind}_H^G(C')$ as G-algebras.*
2. *If C is an H-subalgebra of C', then $\mathrm{Ind}_H^G(C)$ is a G-subalgebra of $\mathrm{Ind}_H^G(C')$, and $C = C'$ if and only if $\mathrm{Ind}_H^G(C) = \mathrm{Ind}_H^G(C')$.*
3. *If $H \le K \le G$, then $\mathrm{Ind}_K^G(\mathrm{Ind}_H^K(C)) = \mathrm{Ind}_H^G(C)$.*

We now return to our construction of invariant subalgebras. Suppose that $V = \mathrm{Ind}_H^G(W)$, where W is a D-module which is a projective representation of H. The cocycle defining ρ_W is just the restriction of α to $h \times H$. It is automatic that W is irreducible. Since W is a direct summand of $V_H \overset{\text{def}}{=} \mathrm{Res}_H^G(V)$, H must act on W by D-linear automorphisms; this means that W is an $F^\alpha H \otimes D$-module. Note that the induced representation V comes equipped with a distinguished choice of $F^\alpha H$-submodule isomorphic to W (namely $\overline{1} \otimes W$), and the invariant subalgebra we construct below depends on this choice. For ease of notation, we view W as this fixed H-submodule of V. Let $T = \{g_1 = 1, g_2, \dots, g_l\}$ be a left transversal of H, and set $W_i = \overline{g_i} \otimes W$, a D-subspace of $V = F^\alpha G \otimes_{F^\alpha H} W$. Define a map $\Psi_{(H,W,T)} : \mathrm{Ind}_H^G(\mathrm{End}_D(W)) \to A = \mathrm{End}_D(V)$ via the formula $\Psi_{(H,W,T)}((g_i \otimes f))(\overline{g_j} \otimes w) = \delta_{ij}\overline{g_i} \otimes f(w)$ for $f \in \mathrm{End}_D(W)$, $w \in W$ and extending by linearity.

Lemma 3.11. *The map $\Psi_{(H,W)} = \Psi_{(H,W,T)}$ is independent of the choice of transversal. It is an injective G-algebra homomorphism whose image is the block-diagonal subalgebra $\oplus_{i=1}^{l} End_D(W_i)$. In particular, this subalgebra is G-invariant.*

Proof. Let $\Psi = \Psi_{(H,W,T)}$. It is easy to see that Ψ is an embedding of algebras with the specified image, so we need only check that Ψ is an intertwining map. Fix $g \in G$. There exists a permutation $\sigma = \sigma_g \in S_l$ and elements $h_i \in H$ such that $gg_i = g_{\sigma(i)}h_{\sigma(i)}$ for all i. First note that

$$\begin{aligned}\Psi(g \cdot (g_i \otimes f))(\overline{g_j} \otimes w) = \Psi(gg_i \otimes f)(\overline{g_j} \otimes w) &= \Psi(g_{\sigma(i)} \otimes h_{\sigma(i)} \cdot f)(\overline{g_j} \otimes w)\\ &= \delta_{\sigma(i)j}\overline{g_j} \otimes (h_j \cdot f)(w).\end{aligned}$$

On the other hand, a similar calculation using the definition of multiplication in $F^\alpha G$ gives

$$(g \cdot \Psi(g_i \otimes f))(\overline{g_j} \otimes w) = \delta_{\sigma(i)j}\beta\overline{g_j} \otimes \overline{h_j}f(\overline{h_j}^{-1}w),$$

where

$$\beta = \alpha(g_j, h_j)^{-1}\alpha(h_j, h_j^{-1})\alpha(g, g^{-1}g_jh_j)\alpha(g^{-1}g_jh_j, h_j^{-1})^{-1}\alpha(g^{-1}, g_j)\alpha(g, g^{-1})^{-1}.$$

Applying the cocycle condition and the fact that $\alpha(x, 1) = 1 = \alpha(1, x)$ for all $x \in G$, we get

$$\begin{aligned}\beta &= \alpha(g_j, h_j)^{-1}\alpha(h_j, h_j^{-1})\alpha(g_jh_j, h_j^{-1})^{-1}\alpha(g, g^{-1}g_j)\alpha(g^{-1}, g_j)\alpha(g, g^{-1})^{-1}\\ &= \alpha(g_j, h_j)^{-1}\alpha(h_j, h_j^{-1})\alpha(g_jh_j, h_j^{-1})^{-1} = 1,\end{aligned}$$

as desired.

The verification that Ψ does not depend on the transversal is similar, but easier. ∎

Let C be an invariant subalgebra of the H-algebra $\mathrm{End}_D(W)$. It now follows from Proposition 3.10 and the lemma that $\Psi(\mathrm{Ind}_H^G(C))$ is a G-invariant subalgebra of $A = \mathrm{End}_D(V)$. More precisely,

Proposition 3.12. *The map $C \mapsto \Theta_{(H,W,C)} \stackrel{def}{=} \Psi_{(H,W)}(Ind_H^G(C))$ defines an injective lattice homomorphism from the H-invariant subalgebras of $End_D(W)$ to the G-invariant subalgebras of $\oplus_{i=1}^{l} End_D(W_i) \subset A$.*

It is not true that an invariant subalgebra of A can be expressed uniquely in terms of this construction if the initial data (namely H, W, and C) are allowed to vary. Indeed, conjugate data (i.e. gHg^{-1}, $\bar{g}W$, and $g \cdot C \subseteq \mathrm{End}_D(\bar{g}W)$ for some $g \in G$) produces the same invariant subalgebra. However, we will see below that uniqueness does hold if we restrict ourselves to conjugacy classes of initial data with C simple.

In order to show that this construction gives rise to all invariant subalgebras, we first need to associate a transitive permutation representation of G to any invariant subalgebra B. By 3.1, we can write $B = B_1 \oplus \cdots \oplus B_l$, where the B_i are simple. The restriction of the G-action π to B gives rise to a permutation representation of G on the set of B_i's because the algebra automorphism $\pi(g)$ must permute the minimal two-sided ideals of B. More explicitly, let $X = \{e_1, \ldots, e_l\}$ with $e_i = 1_{B_i}$ be the set of central primitive idempotents of B. Since e_i is the unique nonzero idempotent in the center of B_i, it is clear that if $\pi(g)(B_i) = B_j$, then $\pi(g)(e_i) = e_j$. We thus obtain a homomorphism $\bar{\pi}_B : G \to S_l$, where we have identified $S(X)$ with S_l in the obvious way. Note that X is also the set of central primitive idempotents

in $Z_A(B)$ and $Z(B)$. Accordingly, dual pairs give rise to the same permutation representation, as do any invariant subalgebras with the same center.

The permutation representation $\bar{\pi}_B$ can also be defined in terms of the B-isotypic components of V. Recall that $V = \oplus_{i=1}^l V_i$ where $V_i = V_i'$ is the isotypic B-submodule of V corresponding to B_i. Fix $g \in G$ and $v \in V_j$, and, write $\bar{g}^{-1}(v) = \sum_{i=1}^l v_i^g$ with $v_i^g \in V_i$. Note that $\bar{g}^{-1}(g \cdot e_i)(v) = e_i(\bar{g}^{-1}(v) = v_i^g$. But by definition, $g \cdot e_i = e_{\bar{\pi}_B(g)(i)}$, giving $v_i^g = \bar{g}^{-1}(g \cdot e_i)(v) = \delta_{j,\bar{\pi}_B(g)(i)}\bar{g}^{-1}(v) = \delta_{i,\bar{\pi}_B(g^{-1})(j)}\bar{g}^{-1}(v)$. This implies that $\bar{g}^{-1}(V_j) \subseteq V_{\bar{\pi}_B(g^{-1})(j)}$ for all j. Applying this to g^{-1} (or using the fact that $\rho(g)^{-1}$ is surjective) gives the reverse inclusion. Thus, G permutes the V_i's, and this permutation is just $\bar{\pi}_B$.

Proposition 3.13. *The permutation representation $\bar{\pi}_B$ is transitive.*

Proof. Let U be a simple B-submodule of V isomorphic to a minimal left ideal of B_1. By definition, e_1 is the identity map on U. Let e_i be any central primitive idempotent, and choose $g \in G$ such that $\bar{g}U$ is a simple B_i module. For all $u \in U$, we have $(g \cdot e_1)(\bar{g}u) = \bar{g}(e_1(\bar{g}^{-1}(\bar{g}u) = \bar{g}(e_1(u)) = \bar{g}(u)$. Since e_i is the unique central primitive idempotent acting as the identity on $\bar{g}U$, this implies that $g \cdot e_1 = e_i$. $\square$

If G acts on B by inner automorphisms, then the G-action preserves the simple components of B. We thus obtain the useful corollary:

Corollary 3.14. *If G acts on the invariant subalgebra B by inner automorphisms, then B is simple.*

Let $H_i = \{g \in G \mid g \cdot e_i = e_i\}$ be the inertia subgroup of e_i. Note that it has finite index l in G. It is immediate that V_i is an $F^\alpha H_i \otimes D$ submodule of V, and the transitivity of $\bar{\pi}_B$ implies that $V = \mathrm{Ind}_{H_i}^G(V_i)$, i.e. V is isomorphic to the induced representation and has distinguished H_i-submodule V_i. Moreover, V_i is an (F-)irreducible projective representation of H_i because if M were a proper subrepresentation, then $\mathrm{Ind}_{H_i}^G(M)$ would be a proper G-submodule of V, contradicting the irreducibility of V. The algebra B_i is a simple H_i-subalgebra of $\mathrm{End}_D(V_i)$, and we are precisely in the situation of the fundamental construction. The uniqueness part of Proposition 3.9 shows that $B = \Theta_{(H_i,V_i,B_i)}$. We have thus realized B in l different ways, all of which have conjugate initial data.

Now suppose that $B = \Theta_{(H,W,C)}$. By definition, W is the isotypic B-submodule corresponding to the simple component C (i.e. $1 \otimes C$) of $B = \mathrm{Ind}_H^G(C)$, implying that $W = V_j$ and $C = B_j$ for some j. Also, H is the stabilizer of B_j, so in fact $H = H_j$.

Let $\mathcal{D}$ be the set of equivalence classes of triples (H, W, C) where H is a subgroup of finite index in G, W is an $F^\alpha H \otimes D$ submodule of V such that $V \cong \mathrm{Ind}_H^G(W)$, and C is an invariant subalgebra of the H-algebra $\mathrm{End}_D(W)$. Also, let $\mathcal{D}_{(H,W)} \subset \mathcal{D}$ be the subset of classes with a representative of the form (H, W, C).

Theorem 3.15. *Let $A = End_D(V)$ be a G-simple central simple algebra. The map $(H, W, C) \mapsto \Theta_{(H,W,C)}$ gives a bijective correspondence between $\mathcal{D}$ and the set of unital G-invariant subalgebras of A. This bijection preserves dual pairs and centers; if $B = \Theta_{(H,W,C)}$, then $Z_A(B) = \Theta_{(H,W,Z_{End_D(W)}(C))}$ and $Z(B) = \Theta_{(H,W,Z(C))}$. Similarly, $Z_0(B)$ (the F-linear span of the Wedderburn components of B) is just $\Theta_{(H,W,Z_0(C))} = \Theta_{(H,W,F1_{End_D(W)})}$. Furthermore, the image of $\mathcal{D}_{(H,W)}$ under the correspondence is precisely the set of invariant subalgebras B with $Z_0(B) = \Theta_{(H,W,F1_{End_D(W)})}$.*

Proof. We have already shown that there is a bijection between invariant subalgebras and triples (H, W, C) where W is an $F^\alpha H$-submodule of V such that the obvious map $W \to \overline{1} \otimes W$ extends to an isomorphism $V \cong \mathrm{Ind}_H^G(W)$. This amounts to saying that V is the internal direct sum of the translates $\overline{g_i}W$ (and so V can be viewed as equal and not just isomorphic to $\mathrm{Ind}_H^G(W)$). The following lemma shows that any subrepresentation of V_H isomorphic to W satisfies this condition.

Lemma 3.16. *Suppose that $V = Ind_H^G(W)$ with V irreducible. Then if W' is any subrepresentation of V_H isomorphic to W, V is the internal direct sum of the $\overline{g_i}W'$'s.*

Proof. By Frobenius reciprocity, there is a linear isomorphism $\mathrm{Hom}_{F^\alpha H}(W, V_H) \cong \mathrm{Hom}_{F^\alpha G}(V, V)$ given by $f \mapsto \hat{f}$, with $\hat{f}(\overline{g_i} \otimes w) = \overline{g_i} f(w)$. Let $f : W \to V_H$ be an H-map with image W'. Since V is irreducible, $\hat{f}$ is an isomorphism. Accordingly, V is the direct sum of the distinct G-translates of $f(W) = W'$. □

It only remains to prove the last three statements. We have shown that as an F-algebra, $\Theta_{(H,W,C)}$ is just $C^{[G:H]}$ embedded in the block diagonal subalgebra $\oplus_{i=1}^l \mathrm{End}_D(W_i) \subset A$. Since taking finite direct sums commutes with taking dual pairs, centers, and Z_0, the result follows. □

Remarks. 1. Since an invariant subalgebra B can always be expressed trivially as $\Theta_{(G,V,B)}$, it is clear that a nonsimple B can arise from nonconjugate initial data. The class in $\mathcal{D}$ corresponding to B consists of the triples with minimal H (or W or C).

2. Let F be an infinite field. If $V \cong \mathrm{Ind}_H^G(W)$ and V_H does not have a unique subrepresentation isomorphic to W, then A has an infinite number of invariant subalgebras. Indeed, in this case, the W-isotypic submodule of V_H is a direct sum of $t \geq 2$ submodules isomorphic to W, so there are an infinite number of submodules W' isomorphic to W. At most $[G : H]$ of these submodules can be conjugate, and each class gives rise to a distinct invariant subalgebra $\Theta_{(H,W',F)}$.

Before proceeding, we give two examples in the case $A = \mathrm{End}_F(V)$.

Examples. 1. Let V be primitive, i.e. suppose that V is not induced from any proper subgroup. Then all invariant subalgebras of A are simple.

2. The theorem shows that V is a monomial representation, i.e. it is induced from a linear character, if and only if $\mathrm{End}_F(V)$ has a G-invariant split Cartan subalgebra $\mathfrak{h}$. Indeed, this can be shown directly. By choosing an appropriate basis for V, we can view $\mathfrak{h}$ as the subalgebra of diagonal matrices in $M_n(F)$. Note that for $\mathfrak{h}$ to be G-invariant means precisely that its normalizer $N(\mathfrak{h})$ contains $\rho(G)$. But $N(\mathfrak{h})$ is the set of monomial matrices, and it is well known that V is monomial if and only if $\rho(G)$ consists of monomial matrices with respect to some basis for V. [I, p.67].

The correspondence in this theorem becomes much simpler when V has nice rationality properties. Recall that a projective F-representation V is called absolutely irreducible if $V_E = V \otimes E$ is an irreducible projective E-representation for every algebraic extension E of F. Equivalently, the division algebra $\mathrm{End}_G(V) \stackrel{\mathrm{def}}{=} \mathrm{End}_{F^\alpha G}(V)$ is just the ground field F. Note that if F is algebraically closed, then all irreducible representations are absolutely irreducible.

Lemma 3.17. *Let A be G-simple. If $K = End_G(V)$, then $D = End_A(V) \subseteq K$. In particular, if V is absolutely irreducible, then $D = F$ and $A = End_F(V)$.*

Proof. Choose $d \in D$. Then we have $d(\rho(g)v) = \rho(g)(dv)$ for $g \in G$, $v \in V$, since $\rho(g) \in A$. Hence, $d \in K$. $\square$

If V is absolutely irreducible, we call such $A = \mathrm{End}_F(V)$ absolutely G-simple.

Now suppose that H is a subgroup of finite index and W is an (irreducible) $F^\alpha H$-module such that $V \cong \mathrm{Ind}_H^G(W)$. Here, we are not viewing W as a specific subspace of V. If V is absolutely irreducible, then $\mathrm{Hom}_{F^\alpha G}(\mathrm{Ind}_H^G(W), V)$ is one-dimensional. By Frobenius reciprocity, the same is true for $\mathrm{Hom}_{F^\alpha H}(W, V_H)$. This implies that there is a unique subrepresentation of V_H isomorphic to W, since otherwise there would be linearly independent H-maps $W \to V_H$. Similarly, we must have $\mathrm{End}_H(W) = F$. Summing up:

Proposition 3.18. *Let V be absolutely irreducible, and suppose that $V \cong Ind_H^G(W)$ where H is a subgroup of finite index and W is an irreducible $F^\alpha H$-module. Then there is a unique subrepresentation of V_H isomorphic to W. Moreover, W is absolutely irreducible.*

Let $\tilde{\mathcal{D}}$ be the set of conjugacy classes of triples where W is only defined up to isomorphism, i.e. W is no longer viewed as a specific subspace of V. In other words, $\tilde{\mathcal{D}}$ consists of the classes of $\mathcal{D}$ modulo H-isomorphism of the second variable. It is clear that triples in $\mathcal{D}$ representing the same class in $\tilde{\mathcal{D}}$ give rise to invariant subalgebras that are isomorphic as G-algebras. If V is absolutely irreducible, the previous proposition shows that the projection $\mathcal{D} \to \tilde{\mathcal{D}}$ is a bijection. Accordingly, we get the first statement of the corollary:

Corollary 3.19. *Let $A = End_F(V)$ be absolutely G-simple. The map $(H, W, C) \mapsto \Theta_{(H,W,C)}$ gives a bijective correspondence between $\tilde{\mathcal{D}}$ and the set of unital G-invariant subalgebras of A. In addition, $\Theta_{(H,W,C)}$ is separable; equivalently, $Z(C)$ is a separable field extension of F.*

Proof. Write $B = \Theta_{(H,W,C)}$. Extending scalars to the algebraic extension E gives the invariant subalgebra B_E of the central simple E-algebra $A_E \cong \mathrm{End}_{D_E}(V_E)$. Since V_E is irreducible, Proposition 3.1 applies, showing that B_E is semisimple. Thus, B is separable. $\square$

We are now ready to make the correspondence in Theorem 3.15 entirely explicit when F is algebraically closed. We start by classifying invariant central simple subalgebras of any G-simple A.

Let B be a simple subalgebra of $A = \mathrm{End}_D(V)$ with simple B-module W' and simple $B \otimes D$-module $W = W' \otimes D$. The $B \otimes D$-module V is isotypic, say $V \cong mW$. Let $L = \mathrm{End}_B(W') = \mathrm{End}_{B\otimes D}(W)$ and set $U = (L^{op})^m$. We obtain the factorization $V \cong W \otimes_{L^{op}} U \cong (W' \otimes_{L^{op}} U) \otimes_F D$. As shown in the proof of Theorem 3.5, $Z_A(B) = \mathrm{End}_{L^{op}}(U)$; also, $B \cong \mathrm{End}_L(W') \cong \mathrm{End}_{L\otimes D}(W)$. In addition, any dual pair of simple subalgebras arises in this way.

Proposition 3.20. *Let $A = End_D(V)$ be a central simple algebra. If $V \cong W \otimes_{L^{op}} U \cong (W' \otimes_{L^{op}} U) \otimes_F D$ with W' an L-module, U an L^{op}-module, and $W = W' \otimes D$ an $L \otimes D$-module, then $End_L(W')$ and $End_{L^{op}}(U)$ is a dual pair of simple subalgebras. Conversely, any dual pair of simple subalgebras comes from such a factorization. In addition, the subalgebras are central simple if and only if L is a central division algebra.*

Using this result, we can classify invariant central simple subalgebras. Let L be a central division algebra, and let W' and U be L and L^{op} modules respectively which are projective representations given by $G \overset{\rho_{W'}}{\to} \mathrm{End}_L(W')^\times$ and $G \overset{\rho_U}{\to} \mathrm{End}_{L^{op}}(U)^\times$. Set $V = (W' \otimes_{L^{op}} U) \otimes_F D$, and let τ denote the canonical isomorphism $\mathrm{End}_L(W') \otimes_F \mathrm{End}_{L^{op}}(U) \overset{\tau}{\to} \mathrm{End}_D(V)$ given by $\tau(f_1 \otimes f_2)(w' \otimes u \otimes d) = f_1(w) \otimes f_2(u) \otimes d$. Then $\rho_V : G \to \mathrm{End}_D(V)^\times$ defined by $\rho_V(g) = \tau(\rho_{W'}(g) \otimes \rho_U(g))$ makes V into a projective representation. It is easy to check that τ becomes a G-algebra isomorphism. If $\rho_{W'}$ and ρ_U are twisted by (one-dimensional) projective characters, then the new G-action on V is projectively equivalent to the old one.

Conversely, suppose that V is a projective representation, and $\mathrm{End}_L(W')$ and $\mathrm{End}_{L^{op}}(U)$ are invariant. The map τ is thus a G-algebra isomorphism. By Proposition 2.1, the G-actions on these subalgebras come from projective representations $(W', \rho_{W'})$ and (U, ρ_U). Hence, $\tau^{-1}(\rho_{W'} \otimes \rho_U)$ and ρ_V define the same G-algebra structure on $\mathrm{End}_D(V)$, implying that they are projectively equivalent, i.e. differ by a projective character. Modifying ρ_U by this twist, we get $\rho_V = \tau(\rho_{W'} \otimes \rho_U)$. It is obvious that if V is irreducible, then both W' and U must be as well. This proves the following theorem:

Theorem 3.21. *Let $A = End_D(V)$ be G-simple. Suppose that $V \cong (W' \otimes_{L^{op}} U) \otimes_F D$ is a factorization such that L is a central division algebra and W' and U are (irreducible) projective representations of G (via L and L^{op} linear automorphisms respectively). Then $A \cong End_L(W') \otimes End_{L^{op}}(U)$ as G-algebras and the images of the two factors in A are a dual pair of invariant central simple subalgebras. Conversely, any such dual pair arises in this way.*

Remark. If $D = F$, invariant central simple subalgebras come from expressing V as the tensor product of projective representations. In general, finding all (or even some) factorizations for a given V is a difficult problem. See for example [St].

We can say more when V is absolutely irreducible. Recall that in this case, $D = F$ and $W = W' \otimes_F D = W'$. Since $\mathrm{End}_{F^\alpha G}(V) = F$, any two G-maps $W \otimes_{L^{op}} U \overset{\sim}{\to} V$ are scalar multiples of each other and thus give the same dual pair of invariant central simple subalgebras. Thus, the specific factorization does not matter.

Corollary 3.22. *If A is absolutely G-simple, then there is a one-to-one correspondence between pairs of irreducible projective representations (W, U) modulo projective equivalence such that $V \cong (W \otimes_{L^{op}} U)$ and dual pairs of invariant central simple subalgebras.*

We now describe the index set for the classification of invariant subalgebras in the algebraically closed case. Let $\mathcal{E}'$ be the set of quadruples (H, W, W_1, W_2) where H is a subgroup of G of finite index, W is an irreducible projective representation of H such that $V \cong \mathrm{Ind}_H^G(W)$, and W_1, and W_2 are irreducible projective representations of H such that $W \cong W_1 \otimes_F W_2$. We then let $\mathcal{E}$ be the set of equivalence classes of $\mathcal{E}'$ where two quadruples (H, W, W_1, W_2) and (H', W', W_1', W_2') are equivalent if there exists $g \in G$ such that $H' = H^g$, $W' = W^g$, and W_i' is projectively equivalent to W_i^g. We let $\mathcal{E}_{(H,W)} \subset \mathcal{E}$ be the subset of classes with a representative of the form (H, W, W_1, W_2). In addition, we denote by $C(W_1, W_2)$ the image of $\mathrm{End}_F(W_1) \otimes 1$ under the isomorphism $\mathrm{End}_F(W_1) \otimes \mathrm{End}_F(W_2) \to \mathrm{End}_F(W)$. The trivial factorizations give $C(F, W) = F$ and $C(W, F) = \mathrm{End}_F(W)$.

Theorem 3.23. *Let F be algebraically closed and $A = End_F(V)$ a G-simple algebra. Then the map $(H, W, W_1, W_2) \mapsto \Theta_{(H,W,C(W_1,W_2))}$ gives a bijective correspondence between $\mathcal{E}$ and the set of invariant subalgebras of A. Moreover, the duality on invariant subalgebras is given by interchanging the W_i's, i.e. $Z_A(\Theta_{(H,W,C(W_1,W_2))}) = \Theta_{(H,W,C(W_2,W_1))}$. The image of $\mathcal{E}_{(H,W)}$ under the correspondence is precisely the set of invariant subalgebras B with center $\Theta_{(H,W,C(F,W))}$.*

Proof. Recall that $\tilde{\mathcal{D}}$ is the set of classes of triples (H, W, C) where H and W are defined as in $\mathcal{E}$ and C is a (central) simple subalgebra of $\mathrm{End}_F(W)$ (using the fact that F is algebraically closed). Since V is absolutely irreducible, Corollary 3.19 shows that invariant subalgebras are parameterized by this set. Applying Corollary 3.22, we see that the map $(H, W, W_1, W_2) \mapsto (H, W, C(W_1, W_2))$ induces a bijection $\mathcal{E} \to \tilde{\mathcal{D}}$, and we obtain the desired correspondence. Since $Z_{\mathrm{End}_F(W)}(C(W_1, W_2)) = C(W_2, W_1)$ and $Z(C(W_1, W_2)) = C(F, W)$, the last statements follow from Theorem 3.15. ∎

Remark. Note that the cocycle α does not determine the cocycles defined by ρ_{W_1} and ρ_{W_2}. In particular, even if V is a linear representation, it is not possible to avoid considering projective representations when studying invariant subalgebras of $\mathrm{End}_F(V)$.

It is convenient to reformulate this correspondence in terms of covering groups. Recall that $\tilde{G}$ is an F^*-generalized covering (or representation) group for G if it is a central extension of G satisfying the projective lifting property for projective representations over F. It is known that F^*-generalized covering groups always exist. If F is algebraically closed and G is finite, then we can choose $\tilde{G}$ finite of order $|G||H^2(G, F^*)|$; such a group is called an F^*-covering group for G [BT].

We now assume that F is algebraically closed (so D and L are just F and $W = W'$). Suppose that the projective representation V factors as $V \cong W \otimes_F U$. Choose a linear representation $(V, \tilde{\rho}_V)$ of $\tilde{G}$ lifting ρ_V and similarly for W and U. A priori, V is only projectively equivalent to $W \otimes U$ over $\tilde{G}$. However, if V_1 and V_2 are linear representations which are projectively equivalent, then $V_1 \cong V_2 \otimes \lambda$, where λ is a linear character. Thus, by choosing a different lift for ρ_W, we obtain linear representations of $\tilde{G}$ such that $V \cong W \otimes_F U$ as $\tilde{G}$-modules. On the other hand, it is obvious that any such factorization gives an isomorphism of projective representations for G.

This allows us to redefine $\mathcal{E}_{(H,W)}$. Let $\tilde{H}$ be a generalized covering group for H, and fix a lift of W to a linear representation of $\tilde{H}$. If (W_1, W_2) and (W_1', W_2') are two pairs of linear representations of $\tilde{H}$ satisfying $W \cong W_1 \otimes W_2 \cong W_1' \otimes W_2'$, we say they are equivalent if for some linear character λ of $\tilde{H}$, $W_1' \cong W_1 \otimes \lambda$ and $W_1' \cong W_1 \otimes \lambda^{-1}$. Denote the set of such classes by $\mathcal{F}_{(H,W)}$. The previous observations give the following result.

Lemma 3.24. *There is a natural bijection between $\mathcal{E}_{(H,W)}$ and $\mathcal{F}_{(H,W)}$.*

Let Y be a complete set of representatives of the conjugacy classes of pairs (H, W). Then the $\mathcal{E}_y$'s partition $\mathcal{E}$. Set $\mathcal{F} = \coprod_{y \in Y} \mathcal{F}_y$. We can now rewrite Theorem 3.23.

Theorem 3.25. *Let F be algebraically closed and $A = End_F(V)$ a G-simple algebra. Then the map $(H, W, W_1, W_2) \mapsto \Theta_{(H,W,C(W_1,W_2))}$ gives a bijective correspondence between $\mathcal{F}$ and the set of invariant subalgebras of A. Duals and centers of invariant subalgebras are given by the same formulas as before.*

It is possible to avoid all explicit mention of projective representations in classifying invariant subalgebras. In order to do this, choose a generalized covering group $\tilde{G}$ of G and fix a lift of V to a representation of $\tilde{G}$. Since the G and $\tilde{G}$ invariant subspaces of A are the same, we can apply the above procedure to the $\tilde{G}$-simple algebra A. Note that this will require choosing a generalized covering group $\tilde{\tilde{G}}$ of $\tilde{G}$!

If F is not algebraically closed, it is not true in general that a simple G-algebra A will have a finite number of invariant subalgebras, even when G is finite. We have already seen a way that finiteness can fail if F is infinite and V is not absolutely irreducible. Namely, if $V \cong \mathrm{Ind}_H^G(W)$ and V_H does not have a unique subrepresentation isomorphic to W, then for any simple H-invariant $C \subset \mathrm{End}_D(W)$, the set $\{\Theta_{(H,W',C)} \mid W' \subset V,\ W' \cong W\}$ will be infinite. Note that these subalgebras are all nonsimple.

Furthermore, the set of invariant subalgebras can be infinite even when V is primitive. Indeed, we have the proposition:

Proposition 3.26. *Let $A = End_F(V)$ where V is an irreducible projective representation of G, and suppose that the division algebra $End_G(V)$ is not a field. Then $\mathcal{D}_{(G,V)}$ is infinite, i.e $End_F(V)$ has an infinite number of simple invariant subalgebras.*

Proof. Note that any subalgebra of $\mathrm{End}_G(V) = (\mathrm{End}_F(V))^G$ is G-invariant, so the following lemma gives the result. $\square$

Lemma 3.27. *Let D be a noncommutative central F-division algebra. Then D contains an infinite number of distinct subfields.*

Proof. Choose noncommuting elements $u, v \in D$, and consider the subfields $F_a = F(u + av)$ for $a \in F$. Wedderburn's theorem on finite division rings shows that the field F is infinite, so it suffices to show that $F_a = F_b$ if and only if $a = b$. If $F_a = F_b$, then $u + av$ and $u + bv$ commute, implying that $auv + bvu = buv + avu$. If uv and vu are linearly independent over F, it is immediate that $a = b$. Otherwise, $vu = cuv$ for some $c \in F$, giving $(a + bc)uv = (b + ac)uv$ and $(c - 1)(a - b) = 0$. Since $c \neq 1$, $a = b$. $\square$

However, these pathologies cannot occur when F is algebraically closed.

Theorem 3.28. *Let F be algebraically closed, G a finite group, and $A = End_F(V)$ a G-simple algebra. Then A has a finite number of invariant subalgebras.*

Proof. Replacing G by a covering group (which is also finite), we can assume without loss of generality that V is a linear representation of G. Since the set of invariant subalgebras and $\mathcal{F} = \coprod_{y \in Y} \mathcal{F}_y$ have the same cardinality (using the notation of Theorem 3.25), it suffices to show that Y and the $\mathcal{F}_y$'s are finite. A theorem of Berman and Witt shows that for arbitrary F, the number of nonisomorphic irreducible F-representations of a finite group is finite [K, Theorem 17.5.3]. The set Y is finite because it is contained in the set of all pairs (H, W) where H is a subgroup

of G and W is an isomorphism class of irreducible FH-modules. Also, $\mathcal{F}_{(H_y,W_y)}$ is finite, since it is smaller than the set of arbitrary pairs of isomorphism classes of irreducible $F\tilde{H}$-modules, where $\tilde{H}$ is a covering group for H. □

We conclude this section with an application to nonunital invariant subalgebras.

Proposition 3.29. *Let F be an algebraically closed field and V an irreducible primitive projective representation of G. Then $\{0\}$ is the only nonunital invariant subalgebra of $A = End_F(V)$. Equivalently, any nonzero subrepresentation of A closed under multiplication must contain the identity.*

Proof. We begin with a lemma.

Lemma 3.30. *Let F be an algebraically closed field. For $t \geq 2$, the matrix algebra $M_t(F)$ has no nonunital subalgebras of codimension one.*

Proof. Suppose that Q is a nonunital subalgebra of codimension one. First note that any element of Q must be singular. To see this, take $q \in Q$ invertible, so that $\det q \neq 0$. It is a well-known corollary of the Cayley-Hamilton theorem that q^{-1} can be expressed as a polynomial in q, so $q^{-1} \in Q$. This implies that Q contains the identity, a contradiction. Thus, $Q \subseteq V(\det)$, the hypersurface of $M_t(F)$ cut out by the determinant. But Q is also a codimension one linear subvariety, so $Q = V(f)$ for some homogeneous degree one polynomial f. As a result, f divides det, and this cannot be true, since the determinant is an irreducible polynomial of degree t. □

Now, let Q be an nonunital invariant subalgebra. Then $Q' = Q + F1_A$ is a unital invariant subalgebra. We know from the first example after Theorem 3.15 that Q' is simple, hence isomorphic to $M_t(F)$ for some $t \geq 1$. If $t = 1$, then $Q = \{0\}$. Applying the lemma finishes the proof. □

4. Invariant subalgebras for topological and Lie groups

In this section, we classify invariant subalgebras in the case where V is a continuous irreducible complex projective representation of a compact connected Lie group. For the moment, we consider a more general situation. Suppose that G is a topological group, $A = \mathrm{End}_D(V)$ is a G-simple algebra endowed with a T_1 topology, and G acts continuously on A. For example, the topology on A could come from F having the structure of a T_1 topological field or $\mathrm{End}_F(V)$ could be given the Zariski topology. So far, this setting includes every abstract group G and G-algebra considered in the previous section by giving G and A the discrete topology. In order to avoid this type of triviality, we further assume that the connected component of the identity G^o (a closed normal subgroup) acts irreducibly on V. We call such an algebra topologically G-simple.

Proposition 4.1. *Every invariant subalgebra of a topologically G-simple algebra A is simple.*

Proof. A G-invariant algebra is also G^o-invariant, so it suffices to assume that G is connected. Let X be the set of central primitive idempotents of an invariant subalgebra B. The transitivity of π_b shows that X is connected. However, since A is T_1, X is discrete. This implies that X is a singleton, i.e. B is simple. □

If we further assume that F is algebraically closed, Theorem 3.22 now applies to give a classification of the invariant subalgebras of $A = \mathrm{End}_F(V)$ in terms of factorizations $V \cong W_1 \otimes W_2$ modulo projective equivalence.

We now assume that $F = \mathbf{C}$ and G is a compact Lie group. Note that a continuous homomorphism $G \to \mathrm{Aut}_{F-\mathrm{alg}}(A) \subset GL(A)$ is a continuous homomorphism $G \to PGL(V)$. Thus, if A is a continuous G-algebra, then V is a continuous projective representation.

Lemma 4.2. *Suppose that G is a simple compact connected Lie group, and let $V(\lambda)$ and $V(\mu)$ be irreducible representations with highest weights λ and μ. Then $V(\lambda) \otimes V(\mu)$ is irreducible if and only if λ or μ is 0.*

Proof. Since $V(\lambda+\mu)$ is a component of $V(\lambda) \otimes V(\mu)$, it suffices to compare the dimension of these representations. The Weyl dimension formula states that

$$\dim V(\lambda) = \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle},$$

where R^+ is the set of positive roots, ρ is half the sum of the positive roots, and $\langle\ ,\ \rangle$ is the Killing form. The equation $\langle \alpha, \lambda+\mu+\rho\rangle\langle\alpha,\rho\rangle + \langle\alpha,\lambda\rangle\langle\alpha,\mu\rangle = \langle\alpha,\lambda+\rho\rangle\langle\alpha,\mu+\rho\rangle$ shows that

$$\frac{\langle \alpha, \lambda+\mu+\rho\rangle}{\langle \alpha,\rho\rangle} \leq \frac{\langle \alpha, \lambda+\rho\rangle}{\langle \alpha,\rho\rangle}\frac{\langle \alpha, \mu+\rho\rangle}{\langle \alpha,\rho\rangle},$$

with equality if and only if $\langle\alpha,\lambda\rangle\langle\alpha,\mu\rangle = 0$. If β is the highest root, then $\langle\beta,\nu\rangle > 0$ for any nonzero dominant weight ν. Multiplying over all positive roots, it follows easily that $\dim V(\lambda+\mu) < \dim V(\lambda)\dim V(\mu)$ if and only if both λ and μ are nonzero. ☐

Let G be a compact connected Lie group. It is well known that the universal covering group of G is of the form $\tilde{G} = G_1 \times \cdots \times G_s \times \mathbf{R}^n$, where each G_i is a simple, simply connected, compact Lie group. Let V be an irreducible projective representation of G. Then V can be lifted to an irreducible representation of $\tilde{G}$, which can be expressed as $V_1 \otimes \ldots \otimes V_s \otimes L$, where V_i is a complex irreducible representation of G_i and L is a character of $\mathbf{R}^n$. This means that V is projectively equivalent to $\tilde{V} = V_1 \otimes \ldots \otimes V_s$. Moreover, simple Lie groups have no nontrivial characters, so projective and linear equivalence are the same for representations of $G_1 \times \cdots \times G_s$. The lemma shows that any factorization of $\tilde{V} = W \otimes W'$ into the tensor product of two representations of $\tilde{G}$ must have W and W' as complementary partial products of $V_1 \otimes \ldots \otimes V_s$. More precisely, let $I = \{i \mid V_i \neq \mathbf{C}\}$ and take $J \subset I$. Set $W_J = \bigotimes_{i=1}^s W_{Ji}$ and $W'_J = \bigotimes_{i=1}^s W'_{Ji}$, where W_{Ji} is V_i if $i \in J$ and $\mathbf{C}$ otherwise and W'_{Ji} is V_i if $i \neq J$ and $\mathbf{C}$ otherwise. We get a factorization $\tilde{V} = W_J \otimes W'_J$, and $J \mapsto W_J$ gives a one-to-one correspondence between the subsets of I and the factors of $\tilde{V}$. This observation combined with Theorem 3.22 proves the following theorem due to Etingof:

Theorem 4.3. *Let G be a compact connected Lie group, and let $A = End_{\mathbf{C}}(V)$ where V is an irreducible projective representation of G projectively equivalent to $V_1 \otimes \ldots \otimes V_s$. Then there is a bijective correspondence between $\mathcal{P}(I)$, the power set of $I = \{i \mid V_i \neq \mathbf{C}\}$, and the set of invariant subalgebras of A, given by $J \mapsto End_{\mathbf{C}}(W_J)$. Moreover, the duality operator corresponds to taking complements in $\mathcal{P}(I)$, i.e. it is given by $End_{\mathbf{C}}(W_J) \mapsto End_{\mathbf{C}}(W_{I-J})$.*

By Theorem 3.22, there are no nontrivial invariant subalgebras not containing 1_A, so we obtain the corollary:

Corollary 4.4. *There are exactly* $2^{|I|}+1$ *subrepresentations of* $\mathrm{End}_{\mathbf{C}}(V)$ *which are closed under matrix multiplication:* $2^{|I|}$ *unital subalgebras and* $\{0\}$.

In particular, if G is a simple compact connected Lie group, then no topologically G-simple algebra has any nontrivial invariant subalgebras. It would be interesting to find classes of finite group satisfying this property and to find a group-theoretic characterization of such groups. It is not true that finite simple groups have this property. In the notation of the atlas of finite groups, $U_4(2)$ has irreducible representations χ_3 and χ_4 of dimensions five and six respectively such that $\chi_3 \otimes \chi_4 \cong \chi_{12}$ is also irreducible [C].

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